1.(6pts) Find symmetric equations of the line L passing through the point $(2, -5, 1)$ and perpendicular to the plane $x + 3y - z = 9$.

(a) $\dfrac{x-2}{1} = \dfrac{y+5}{3} = \dfrac{z-1}{-1}$

(b) $\dfrac{x-1}{2} = \dfrac{y-3}{-5} = \dfrac{z+1}{1}$

(c) $2(x-1) = (-5)(y-3) = z+1$

(d) $\dfrac{x-1}{2} = \dfrac{y-3}{-5} = \dfrac{z+1}{1} = 9$

(e) $(x-2) + 3(y-3) - (z-1) = 9$

Solution. The symmetric equations of line are given by $(x - x_0)/a = (y - y_0)/b = (z - z_0)/c$, where (x_0, y_0, z_0) is a point on the line and $\langle a, b, c\rangle$ is a direction vector. Since L is perpendicular to the plane $x + 3y - z = 9$, then we can take the normal to the plane as the direction vector, this is, $\langle 1, 3, -1\rangle$ is a direction vector of L. Therefore, the symmetric equations are $\dfrac{x-2}{1} = \dfrac{y+5}{3} = \dfrac{z-1}{-1}$.

2.(6pts) The two curves below intersect at the point $(1, 4, -1) = \mathbf{r}_1(0) = \mathbf{r}_2(1)$. Find the cosine of the angle of intersection

$$\mathbf{r}_1(t) = e^{3t}\mathbf{i} + 4\sin\left(t + \frac{\pi}{2}\right)\mathbf{j} + (t^2 - 1)\mathbf{k}$$
$$\mathbf{r}_2(t) = t\mathbf{i} + 4\mathbf{j} + (t^2 - 2)\mathbf{k}$$

(a) $\dfrac{1}{\sqrt{5}}$ (b) 0 (c) $\dfrac{1}{5}$ (d) $\dfrac{e}{\sqrt{e^2+4}}$ (e) 3

Solution. Note

$$\mathbf{r}_1'(t) = \left\langle 3e^{3t}, 4\cos\left(t + \frac{\pi}{2}\right), 2t\right\rangle$$
$$\mathbf{r}_2'(t) = \langle 1, 0, 2t\rangle$$

To compute the angle of intersection we find $\mathbf{r}_1'(0) = \langle 3, 0, 0\rangle$ $\mathbf{r}_2'(1) = \langle 1, 0, 2\rangle$ so that $\cos\theta = \dfrac{\mathbf{r}_1'(0) \bullet \mathbf{r}_2'(1)}{|\mathbf{r}_1'(0)||\mathbf{r}_2'(1)|} = \dfrac{3}{3\sqrt{5}} = \dfrac{1}{\sqrt{5}}$.

3.(6pts) Find the projection of the vector $\langle 1, -1, 5\rangle$ onto the vector $\langle 2, 1, 4\rangle$

(a) $\langle 2, 1, 4\rangle$ (b) $\langle 1, -1, 5\rangle$ (c) $\langle 6, 3, 12\rangle$ (d) $\langle 3, -3, 15\rangle$ (e) $\dfrac{1}{5}\langle 2, 1, 5\rangle$

Solution.

$$\text{proj}_{\langle 2,1,4\rangle}\big(\langle 1, -1, 5\rangle\big) = \frac{\langle 2, 1, 4\rangle \bullet \langle 1, -1, 5\rangle}{\langle 2, 1, 4\rangle \bullet \langle 2, 1, 4\rangle}\langle 2, 1, 4\rangle = \frac{21}{21}\langle 2, 1, 4\rangle = \langle 2, 1, 4\rangle$$

4.(6pts) Find $\int \mathbf{r}(x)dx$ where

$$\mathbf{r}(x) = (\sec^2 x)\mathbf{i} + e^x\mathbf{k}$$

Recall: $\int \sec^2 x\, dx = \tan x + C.$

(a) $(\tan x + C_1)\mathbf{i} + C_2\mathbf{j} + (e^x + C_3)\mathbf{k}$

(b) $(\tan x + C_1)\mathbf{i} + (e^x + C_2)\mathbf{k}$

(c) $(\tan x)\mathbf{i} + e^x\mathbf{k}$

(d) $\tan x + e^x + C$

(e) $(\tan x + C)\mathbf{i} + C\mathbf{j} + (e^x + C)\mathbf{k}$

Solution.

$$\begin{aligned}\int \mathbf{r}(x)dx &= \int \left((\sec^2 x)\mathbf{i} + e^x\mathbf{k}\right) dx \\ &= \left(\int \sec^2 x dx\right)\mathbf{i} + \left(\int 0 dx\right)\mathbf{j} + \left(\int e^x dx\right)\mathbf{k} \\ &= (\tan x + C_1)\,\mathbf{i} + C_2\mathbf{j} + (e^x + C_3)\,\mathbf{k}\end{aligned}$$

5.(6pts)The curvature of the function $y = 2\sin x$ at $x = \frac{\pi}{2}$ is

(a) 2

(b) 0

(c) $\frac{1}{2}$

(d) $\sqrt{2}$

(e) Does not exist.

Solution.
Let $\mathbf{r}(t) = \langle t, 2\sin t, 0\rangle$. Then $\mathbf{r}'(t) = \langle 1, 2\cos t, 0\rangle$, $\mathbf{r}''(t) = \langle 0, -2\sin t, 0\rangle$, $\mathbf{r}'\left(\frac{\pi}{2}\right) = \langle 1, 0, 0\rangle$, and $\mathbf{r}''\left(\frac{\pi}{2}\right) = \langle 0, -2, 0\rangle$. One of the formulas for curvature says that $\kappa = \dfrac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}$.
But $\mathbf{r}' \times \mathbf{r}''$ at $t = \frac{\pi}{2}$ is $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & -2 & 0 \end{vmatrix} = \langle 0, 0, -2\rangle$ and $|\langle 0, 0, -2\rangle| = 2$.

6.(6pts) What is the (approximate) normal component of the acceleration for a car traveling $29\, m/s$ around a curve with curvature $\kappa = 2/31\, m^{-1}$.

(a) $54.3\, m/s^2$

(b) $1.9\, m/s^2$

(c) $449.5\, m/s^2$

(d) $400.2\, m/s^2$

(e) $40.3\, m/s^2$

Solution. Using the formula that $a_N = \kappa v^2$. Plugging in the values from the problem we find that $a_N = 54.3$.

7.(6pts) Find the area of the triangle formed by the three points $(1, 0, 1)$, $(2, 0, 2)$ and $(3, 3, 3)$.

(a) $\frac{3}{2}\sqrt{2}$

(b) 0

(c) 4

(d) $\frac{\sqrt{3}}{2}$

(e) 2.2

Solution. Two vectors which form two sides of the triangle are $\langle 1,0,1\rangle = \langle 2,0,2\rangle - \langle 1,0,1\rangle$ and $\langle 2,3,2\rangle = \langle 3,3,3\rangle - \langle 1,0,1\rangle$. Hence

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 2 & 3 & 2 \end{vmatrix} = \langle 3-0, -(2-2), 3-0\rangle = \langle 3,0,3\rangle$$

The area of the parallelogram is $|\langle 3,0,3\rangle| = \sqrt{9+0+9} = 3\sqrt{2}$ and the area of the triangle is half this.

8.(6pts) Find the volume of the parallelepiped spanned by the three vectors $\langle 1,2,-1\rangle$, $\langle 0,1,2\rangle$ and $\langle 3,2,1\rangle$.

(a) 12 (b) $2\sqrt{3}$ (c) $9\sqrt{2}$ (d) $3\sqrt{2}$ (e) 0

Solution. Answer is the absolute value of the triple product

$$\begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 2 \\ 3 & 2 & 1 \end{vmatrix} = 1\cdot\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} - (2)\cdot\begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} + -1\cdot\begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} = -3+12+3 = 12$$

9.(10pts) Find an equation for the plane which goes through the point $(1,2,5)$ and contains the line $\langle 2,1,-1\rangle\, t + \langle 3,4,1\rangle$.

Solution. One vector in the plane is $\langle 2,1,-1\rangle$. A second is $\langle 3,4,1\rangle - \langle 1,2,5\rangle = \langle 2,2,-4\rangle$. Hence a normal vector for the plane is

$$\mathbf{N} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -4 \\ 2 & 1 & -1 \end{vmatrix} = \langle -2+4, -(-2+8), 2-4\rangle = \langle 2,-6,-2\rangle$$

Hence an equation is

$$\langle 2,-6,-2\rangle \bullet \langle x,y,z\rangle = \langle 2,-6,-2\rangle \bullet \langle 1,2,5\rangle = 2-12-10 = -20$$

or

$-2x+6y+2z = 20$ or $-x+3y+z = 10$.

10.(10pts) Let Π be the plane passing through the point $Q = (5,10,-3)$ with normal vector $\mathbf{n} = \langle 3,1,1\rangle$. Let P be the point on Π which is closest to the origin. Let ℓ be the line passing through the origin and P.

Hint: parts (a) and (b) can be answered independently of each other.

(a) Find the co-ordinates of P.
(b) Find a vector equation for ℓ.

Solution. We find the position vector of P as

$$\vec{OP}. = \text{Proj}_{\mathbf{n}}\mathbf{OQ} = \frac{\mathbf{OQ}\bullet\mathbf{n}}{\mathbf{n}\bullet\mathbf{n}}\mathbf{n} = \frac{22}{11}\langle 3,1,1\rangle = \langle 6,2,2\rangle$$

and so $P = (6,2,2)$.

The line ℓ has direction $\mathbf{n}$, and passes through the origin, so has vector equation

$$\langle 3,1,1\rangle \bullet \langle x,y,z\rangle = 0$$

11.(10pts) Suppose the curve C has parametric equations:

$$x(t) = t^3 - t, \quad y(t) = 1 - 2\sqrt{t}, \quad z(t) = t^2 + t$$

(a) Let $P = (0, -1, 2)$. Find t_0 so that $P = (x(t_0), y(t_0), z(t_0))$.

(b) Let $\mathbf{r}(t) = \left\langle t^3 - t, 1 - 2\sqrt{t}, t^2 + t \right\rangle$. Find $\mathbf{r}'(t_0)$, the tangent vector to the above curve C at the point $P = (0, -1, 2)$.

(c) Find the parametric equation for the tangent line to the above curve C at the point $P = (0, -1, 2)$.

Solution.

(a) $0 = t_0^3 - t_0 = t_0(t_0 - 1)(t_0 + 1)$ implies $t = 0, 1$, or -1.
$-1 = 1 - 2\sqrt{t_0}$ implies $2\sqrt{t_0} = 2$ i.e. $t_0 = 1$.

(b) $\mathbf{r}'(t) = \left\langle 3t^2 - 1, -t^{-\frac{1}{2}}, 2t + 1 \right\rangle$
$\mathbf{r}'(1) = \langle 2, -1, 3 \rangle$

(c) Let $\mathbf{v}(t) = \langle 2, -1, 3 \rangle$. Then the vector equation for the tangent line to C at P is given by $t\mathbf{v}(1) + \langle 0, -1, 2 \rangle = \langle 2t, -t, 3t \rangle + \langle 0, -1, 2 \rangle = \langle 2t, -t - 1, 3t + 2 \rangle$.
Then the parametric equation for the tangent line to C at P is given by
$x(t) = 2t, \quad y(t) = -t - 1, \quad z(t) = 3t + 2.$

12.(10pts) Suppose a particle has acceleration function $\mathbf{a}(t) = -\dfrac{1}{(t+1)^2}\mathbf{j} - (\sin t)\mathbf{k}$ for $t \geq 0$. Suppose that the initial position is $\mathbf{r}(0) = \mathbf{k}$ and the initial velocity is $\mathbf{v}(0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

(a) Find the *velocity* function.
(b) Find the *position* function.

Solution. First find the velocity function as the integral of the acceleration function

$$\mathbf{v}(t) = \int \mathbf{a}(t)dt = C_1\mathbf{i} + \left(\frac{1}{t+1} + C_2\right)\mathbf{j} + (\cos t + C_3)\mathbf{k}$$

Next we plug in $t = 0$ and compare with the initial velocity

$$\mathbf{v}(0) = C_1\mathbf{i} + \left(\frac{1}{0+1} + C_2\right)\mathbf{j} + (\cos 0 + C_3)\mathbf{k} = C_1\mathbf{i} + (1 + C_2)\mathbf{j} + (1 + C_3)\mathbf{k}$$

and by comparison to the given initial velocity, $\mathbf{v}(0) = \mathbf{i} + \mathbf{j} + \mathbf{k}$, we have

$$C_1 = 1, C_2 = C_3 = 0$$

So, the velocity function is

$$\mathbf{v}(t) = \mathbf{i} + \frac{1}{t+1}\mathbf{j} + (\cos t)\mathbf{k}$$

The position function is the integral of the velocity function

$$\mathbf{r}(t) = \int \mathbf{v}(t)dt = (t + D_1)\mathbf{i} + (\ln(t+1) + D_2)\mathbf{j} + (\sin t + D_3)\mathbf{k}$$

Next, we plug in $t = 0$ and compare with the initial position

$$\mathbf{r}(0) = (0 + D_1)\mathbf{i} + (\ln(0+1) + D_2)\mathbf{j} + (\sin 0 + D_3)\mathbf{k} = D_1\mathbf{i} + D_2\mathbf{j} + D_3\mathbf{k}$$

and by comparison to the given initial position, $\mathbf{r}(0) = \mathbf{k}$, we have

$$D_1 = D_2 = 0 \qquad D_3 = 1$$

Thus, the position function is

$$\mathbf{r}(t) = t\mathbf{i} + \ln(t+1)\mathbf{j} + 1 + \sin t\mathbf{k}$$

OR using vector arithmetic

$$\mathbf{v}(t) = \int \left\langle 0, -\frac{1}{(t+1)^2}, -\sin t \right\rangle dt = \left\langle 0, \frac{1}{t+1}, \cos t \right\rangle + \mathbf{C}$$

Then $\mathbf{v}(0) = \langle 0, 1, 1 \rangle + \mathbf{C} = \langle 1, 1, 1 \rangle$ so $\mathbf{C} = \langle 1, 0, 0 \rangle$ and

$$\mathbf{v}(t) = \left\langle 1, \frac{1}{t+1}, \cos t \right\rangle$$

Then

$$\mathbf{r}(t) = \int \left\langle 1, \frac{1}{t+1}, \cos t \right\rangle dt = \langle t, \ln|t+1|, \sin t \rangle + \mathbf{D}$$

Then $\mathbf{r}(0) = \langle 0, 0, 0 \rangle + \mathbf{D} = \langle 0, 0, 0 \rangle$ so $\mathbf{D} = \langle 0, 0, 1 \rangle$ and

$$\mathbf{r}(t) = \langle t, \ln|t+1|, 1 + \sin t \rangle$$

13.(10pts) Find an equation for the osculating plane of $\mathbf{r}(t) = \langle t, \cos t, e^t \rangle$ when $t = 0$.

Solution. $\mathbf{r}'(t) = \langle 1, -\sin t, e^t \rangle$, $\mathbf{r}''(t) = \langle 0, -\cos t, e^t \rangle$, $\mathbf{r}(0) = \langle 0, 1, 1 \rangle$, $\mathbf{r}'(0) = \langle 1, 0, 1 \rangle$, and $\mathbf{r}''(0) = \langle 0, -1, 1 \rangle$. The normal vector is $\langle 1, 0, 1 \rangle \times \langle 0, -1, 1 \rangle = \langle 1, -1, -1 \rangle$. So an equation is $(x - 0) - (y - 1) - (z - 1) = 0$ or $x - y - z = -2$.